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STEP WAVES IN FLOWING FILMS

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Abstract: *The axisymmetric flow of a thin layer of viscous fluid down the outer surface of a vertical cylinder in the field of gravity is considered. The process is modelled using the Kapitsa-Shkadov approach. Wave modes with step waves characterized by the presence of a film thickness drop are investigated. It is found that the existence regions of step waves solutions are strips.*

Keywords: *liquid film, step wave, hydrodynamic stability, Kapitsa-Shkadov approach*

1. INTRODUCTION

The application of thin coatings is a difficult problem to model. It is reduced to the consideration of flowing of a thin layer of viscous fluid over a solid surface in the field of gravity. The liquid surface in such flows often has a nonlinear wave structure, in which the surface area of the gas-liquid interface increases, which accelerates physicochemical reactions and heat-mass transfer processes.

Experimental and theoretical study of flowing films was laid down by the works of P.L. Kapitsa [1], which reported the discovery of a new «stable regime consisting of solitary waves». It was possible to solve this problem by methods of mechanics and mathematics. V. Ya. Shkadov constructed a mathematical model of film nonstationary flow, taking into account the nonlinearity of the phenomenon [2]. Solutions in the form of solitary waves were obtained [3]. This allowed to interpret the available experimental data and to chart the way towards new meaningful experiments. A detailed description of the mathematical model, the solution method, analysis of the obtained results and comparison with experimental data are presented in [4-6]. The theory has been widely developed and internationally recognised [7].

The present work is devoted to the theoretical study of wave regimes with step waves characterised by a film thickness difference. The description of such flows is presented in [8,9]. The problem of finding these solutions in a film flowing down the exterior of a vertical cylinder is considered. The regions of existence of step waves are found.

Identification of flow parameters corresponding to different stable wave modes is one of the most important tasks for industry. The obtained knowledge allows controlling films in production and fine-tuning technological processes.

2. EQUATION AND BOUNDARY CONDITIONS

The axisymmetric uncurled flow of a viscous thin layer of liquid on the exterior surface of a vertical cylinder in the field of gravity is considered (Fig. 1). In dimensionless formulation the problem has three parameters. The geometric parameter $\boldsymbol{\varepsilon} = \frac{H}{R}$ is equal to the ratio of the average film thickness H to the cylinder radius R. The parameter $\boldsymbol{\kappa} = \frac{H}{L}$ corresponds to the ratio of the average film thickness H to the characteristic wavelength L. It can be assumed to be small, since thin films are considered. The capillary, gravitational, and viscous forces are characterised by a single dimensionless parameter $\boldsymbol{\delta_\varepsilon}$.

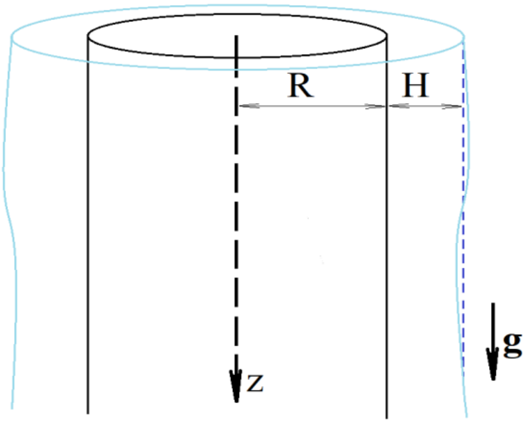

Figure 1. Schematic representation of the flow of a thin layer of liquid with an average thickness H over the outer surface of a vertical cylinder of radius R in the gravity field $\boldsymbol{g}$. The image was made by the authors.

The flow is described in a cylindrical coordinate system. The principal axis $x = \frac{z}{L}$ measured at the characteristic wavelength L is oriented along the flow direction. Let us pass to a mobile coordinate system moving with the wave velocity c: $\xi = x - ct$. The application of the Kapitsa-Shkadov approach reduces the research of the above wave currents to the study of the Shkadov equation (1) for the local layer thickness $h = h(\xi)$ [6]:

$$h^3 h''' + h'\left(h^2 \frac{5\delta_\varepsilon}{1+\frac{\varepsilon h}{2}}(c^2(1+\varepsilon h) - M_h') + \right.$$

$$\left. + \left(\frac{\varepsilon}{\kappa(1+\varepsilon h)}\right)^2 h^3\right) + h^3 - \frac{\phi(\varepsilon)}{\phi(\varepsilon h)} q = 0, \qquad (1)$$

$$q = \frac{c\varepsilon}{2}(h^2 - 1) + c(h - 1) + 1,$$

where $q = q(\xi)$ is the dimensionless local flow rate. We use the notations given in [6].

We call a step wave a solution of equation (1) satisfying the following conditions at infinity:

$$h \to 1 \ at \ \xi \to \mp\infty; \ h \to h_s \ at \ \xi \to \pm\infty \qquad (2)$$

Moreover, h_s is different from 1, i.e., the thickness of the liquid layer changes as the wave passes by.

3. STEP WAVES

In the phase space (h, h', h'') equation (1) has two special points that satisfy the mechanical sense of the problem:

$O_1 = (1, 0, 0)$ and $O_2 = (1 + a, 0, 0)$, $a > -1$

Show the relation between the existence of special points and boundary conditions. For this purpose, we analyze the behavior of the solution near the special points. We linearize equation (1) in the vicinity of the point $(h_i, 0, 0)$ and get this solution:

$$h = h_i + A_{i1} e^{2z_i \xi} + A_{i2} e^{-z_i \xi} \cos(\omega_i \xi - \psi)$$

z_i and w_i are found by Cardan's formulae [8]. A_{i1}, A_{i2} and ψ are unknown constants determined from the solution of the nonlinear problem (1)-(2).

It becomes clear that $h_s = 1 + a$. The existence of two different special points in phase space is the reason why step waves solutions can exist. Thus, a step wave is a heteroclinic trajectory in phase space (h, h', h''), doubly asymptotic to the special points O_1 and O_2. The following cases can be distinguished:

1) the fast negative step wave, $c > \frac{3+\phi_1(\varepsilon,1)}{1+\varepsilon}$ $(z_1 > 0, z_2 < 0, a > 0)$:

$$h \to 1 + A_{11} e^{2z_1 \xi} \ at \ \xi \to -\infty$$

$$h \to 1 + a + A_{21} e^{2z_2 \xi} \ at \ \xi \to +\infty$$

and the fast positive step wave:

$$h \to 1 + a + A_{21} e^{2z_2 \xi} \ at \ \xi \to -\infty$$

$$h \to 1 + A_{11} e^{2z_1 \xi} \ at \ \xi \to +\infty$$

2) the slow negative step wave, $c > \frac{3+\phi_1(\varepsilon,1)}{1+\varepsilon}$

$(z_1 < 0, z_2 > 0, a < 0)$:

$$h \to 1 + a + A_{22} e^{-z_2 \xi} cos(\omega_2 \xi - \psi)$$
$$at\ \xi \to -\infty$$
$$h \to 1 + a + A_{12} e^{-z_1 \xi} cos(\omega_1 \xi - \psi)$$
$$at\ \xi \to +\infty$$

and the slow positive step wave:

$$h \to 1 + A_{12} e^{-z_1 \xi} cos(\omega_1 \xi - \psi)$$
$$at\ \xi \to -\infty$$
$$h \to 1 + a + A_{22} e^{-z_2 \xi} cos(\omega_2 \xi - \psi)$$
$$at\ \xi \to +\infty$$

The meaning of the parameter a is shown: its modulus is nothing but the dimensionless height of the step. A smooth transition between asymptotic solutions damped to 1 and to $1 + a$ at different ends is achieved due to the nonlinearity of equation (1).

For the numerical solution, a Cauchy problem was posed. The initial condition was the value of the asymptotic solution at the left end and the approximate value of the desired parameter of the problem – phase velocity c. Then the system was integrated until the trajectory approached another asymptotic point. The value of the desired parameter was refined using Newton's method due to the boundary condition on the right end.

4. RESULTS

Let us switch to more universal notations. The parameter $\kappa = (45\,\delta\,\gamma^{-3/2})^{2/11}$ can be expressed through the Kapitza number γ. And also, it will be convenient to introduce the parameter $\delta = \frac{\delta_\varepsilon}{9\phi^2(\varepsilon)}$.

For each pair of parameters $(c,\ \delta)$ a sequence of positive step wave families was found. It has some common features with a similar sequence for solitary waves [7]. At parameter values γ=2700, $\varepsilon = 0.1$, $\delta = 0.1$ more than 20 families of fast positive step waves and more than 10 families of slow positive step waves were found. Some of them are shown in the Fig. 2.

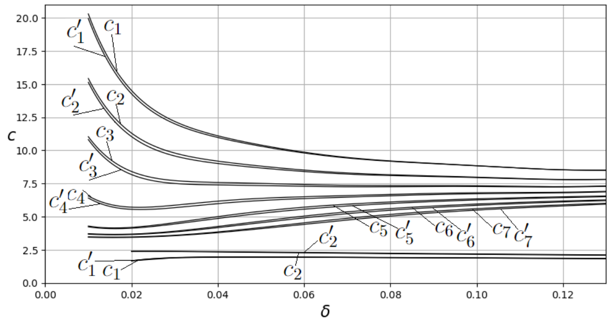

Figure 2. Areas of existence of the first seven families of fast positive step waves: bands (c_1, c'_1), (c_2, c'_2), ..., (c_7, c'_7); and the first two families of slow compression waves: bands (c_1, c'_1), (c_2, c'_2) - these bands practically merge into one line, since the difference in phase velocities for them is of the order of 10^{-3} or less, when γ=2700, $\varepsilon = 0.1$

In the case $c > \frac{3+\phi_1(\varepsilon,1)}{1+\varepsilon}$ (fast waves) each family lies inside the band (c'_n, c_n). The index n corresponds to the family number, with

$$c_1 > c'_1 > c_2 > c'_2 > \cdots > c_n > c'_n.$$

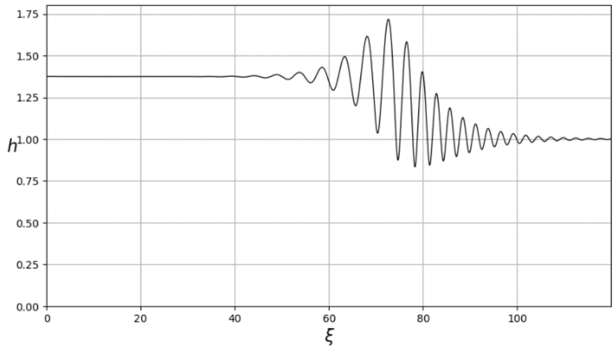

Figure 3. Fast positive step wave: c_{13}: $c = 4.02995, a = -0.375$; γ=2700, $\varepsilon = 0.1$, $\delta = 0.1$

Starting from some $c < c'_n$, the solution degenerates into a flow with constant film thickness equal to $h = 1 + a$ $(a > 0)$ which is explained by the influence of the boundary condition on the left.

At $c < \frac{3+\phi_1(\varepsilon,1)}{1+\varepsilon}$ (slow waves) it is also found that the families are (c_m, c'_m) bands:

$$c'_m > c_m > c'_{m-1} > c_{m-1} > \cdots > c'_1 > c_1.$$

Positive step waves, like solitons, are characterized by the number of principal humps to which in phase space correspond the turns around the unstable equilibrium position: the point O_2 for fast waves and the point O_1 for slow waves. The indices n and m characterise the number of humps – the larger the index, the more humps there are, while the height of the steps, characterized by the value a, on the contrary, decreases. This is explained by the dependence of the step height a on the phase velocity c. When approaching the neutral value of the phase velocity (this is what happens with increasing indices n and m):

$$c \to c_{neutral} = \frac{3 + \phi_1(\varepsilon, 1)}{1 + \varepsilon}$$

a tends to zero. The points O_1 and O_2 get closer together, since the height of the step a actually determines the distance in phase space between these two equilibrium positions.

The trajectories of waves corresponding to the values of c_i' have an additional rotation around two points O_1 and O_2 at once (Fig. 4), which is absent for waves with values c'_i . This is expressed by the complication of the wave structure - an additional main hump appears (Fig. 5). The difference from the regions of existence of soliton solutions – when $\delta \to 0$, the bands of the first families diverge.

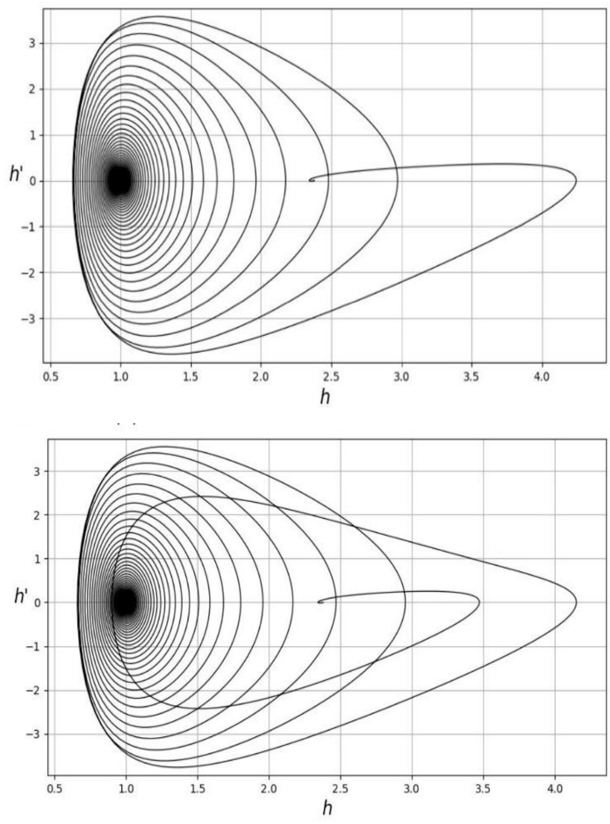

Figure 4. Trajectories of the first family representatives of fast positive step-waves: $c_1 \colon c = 8.821148, a = -1.3777$; $c'_1 \colon c = 8.809395, a = -1.3757$; γ=2700, $\varepsilon = 0.1$, $\delta = 0.1$

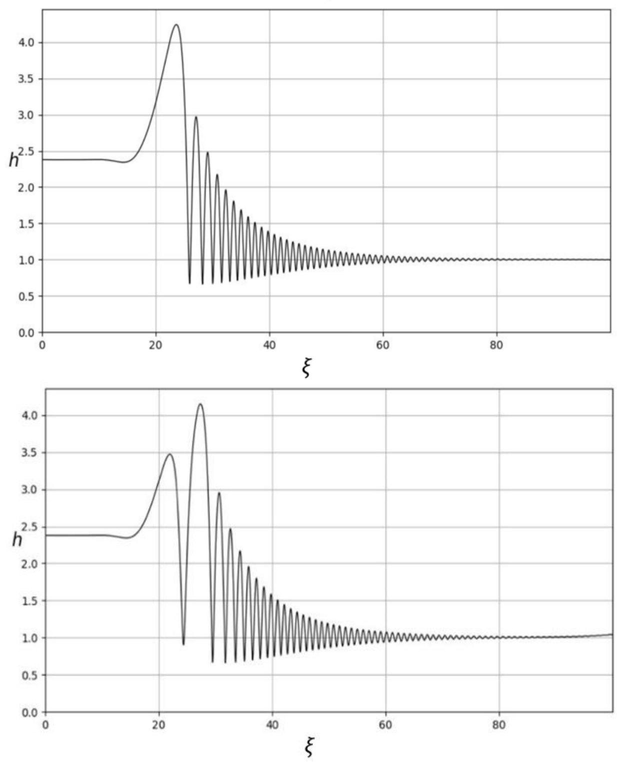

Figure 5. First family representatives of fast positive step waves: $c_1 \colon c = 8.821148, a = -1.3777$; $c'_1 \colon c = 8.809395, a = -1.3757$;

γ=2700, $\varepsilon = 0.1$, $\delta = 0.1$

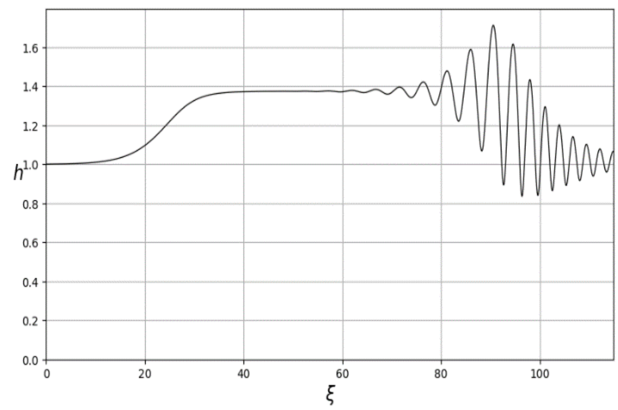

Figure 6. Fast soliton: $c = 4.02995, a = -0.375$; γ=2700, $\varepsilon = 0.1$, $\delta = 0.1$

Two families have also been established for negative step waves: slow and fast. Each corresponds to a continuous spectrum of parameter values c lying within two bands. With the values of the problem parameters $\gamma = 2700$, $\varepsilon = 0.1$, it was found that slow negative step waves lie inside a narrow band $(c_1^s, c_1'^s)$. Further, at c $> c_1'^s$ and up to the value of $c_{neutral}$, they degenerate into a wave-free regime with a film thickness equal to $h = 1 + a$ $(a < 0,\ |a| < 1)$, which is explained by the influence of the left the boundary condition.

The family of fast negative step waves corresponds to a wide band $\left(c_{neutral}, c_1^f\right)$. bordering on the neutral value of the phase velocity. The value of $c = c_1^f$ corresponds to the fastest negative step wave. When the phase velocity decreases to a neutral value, the solution stretches along ξ, and the wave height tends to zero. For the values of $\gamma = 2700$, $\varepsilon = 0.1$, the regions of existence of positive and negative step waves were constructed numerically and shown in Fig. 7.

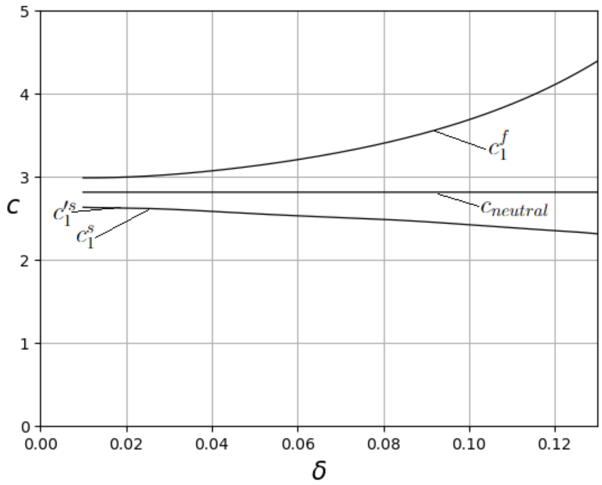

Figure 7. Areas of existence of the negative step waves: the wide band $\left(c_{neutral}, c_1^f\right)$ and the narrow $(c_1^s, c_1'^s)$; γ=2700, $\varepsilon = 0.1$

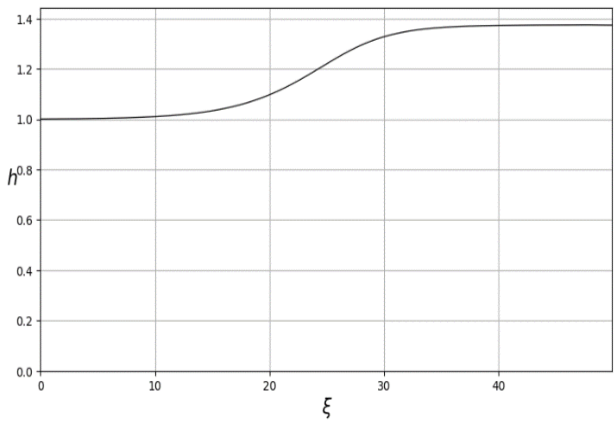

Figure 8. Fast negative step wave: $c = 4.02995, a = 0.375$; γ=2700, $\varepsilon = 0.1$, $\delta = 0.1$

A qualitative analysis of the solutions shows a visual similarity between solutions in the form of fast solitons and positive step waves. Moreover, the phase velocities of some of their representatives differ slightly. Also, for fast solitary waves, at values of the phase velocity c approaching $c_{neutral}$, it can be noted that they become a combination of two step waves: the negative wave smoothly transitions to the positive. This case is demonstrated in Fig. 3, 6, 8 the values of the parameters $\gamma = 2700$, $\varepsilon = 0.1$, $\delta = 0.1$, and $c = 4.02995008$ are the same for these three solutions.

5. CONCLUSION

The obtained results demonstrate the possibility of wide application of the Shkadov equation in film theory. The presented work shows that this approach allows describing various wave modes in film flows, including regimes with step waves.

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